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A GENERALIZED ALLENDOERFFER-WEIL FORMULA AND AN INEQUALITY OF THE COHN-VOSSEN TYPE

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1. Introduction

In this paper we present a version of the Gauss-Bonnet-Chern formula which applies to arbitrary compact locally convex subsets C of a riemannian manifold M. The classical counterpart is the Allendoerffer-Weil formula for riemannian polyhedra [1]. But while the singularities of a polyhedron are separated along submanifolds, we do not have to restrict the set of singularities, in particular, this set can be dense in the boundary of C.

An important role is played in our formula by the set $\mathcal{N}_C$ of outer vectors of C, which is a locally lipschitz submanifold of T_1M. In fact, the boundary terms appear as integrals over $\mathcal{N}_C$ of an almost everywhere defined differential form, in which enter curvature quantities of M along ∂C and a generalized second fundamental form of ∂C which is symmetric and positive semidefinite almost everywhere.

In the case of a positive semidefinite curvature operator along ∂C, the boundary terms can be estimated thus giving a sort of Cohn-Vossen inequality for such sets C. For dimensions not exceeding 6, the assumption on the curvature operator can be replaced by $K \geq 0$ along ∂C. Several corollaries apply to the existence of, and bounds for, the total curvature of complete manifolds of nonnegative curvature.

2. Preliminaries

Let $(M, \langle\ ,\ \rangle)$ always be a smooth oriented and connected riemannian manifold of dimension $m \geq 2$. In any case, all manifolds are supposed to be Hausdorff and paracompact. The general notation for M is the same as in [18].

2.1. If $f: N \to M$ is smooth, by a *form A of bidegree (r, s) along f* we mean an alternating form on N of degree r with values in $\bigwedge^s f^*TM$ [2, § 8.3]. For such forms there are the usual algebraic operations $+$, $\wedge$ and, by the given orientation of M, every form Q of bidegree (r, m) corresponds to a form $[Q]$ of bidegree $(r, 0)$, i.e., to a real r-form on N. If $g: N_1 \to N$ is smooth, then for every given A there is defined a form g^*A of the same bidegree along $f \circ g$. The operation g^* is homomorphic with respect to $+$, $\wedge$, $[\ ]$, and for another

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smooth map $h: N_2 \to N_1$ we have $(g \circ h)^*A = h^*(g^*A)$. Observe that an r-multilinear alternating map from $(T_qN)^r$ into $\bigwedge^s T_{f(q)}M$ is associated with the given A at every $q \in N$.

2.1.1. The connection on f^*TM, induced by the Levi-Civita connection ∇ on M, gives rise to the covariant differential operator D transforming smooth forms of bidegree (r, s) along f into such forms of bidegree $(r + 1, s)$. For $s = m$, D commutes with [ ].

2.1.2. If the manifolds N, N_1, N_2 are of class $\mathscr{C}^1$, and f, g, h are only continuous, then according to the last remark in § 2.1 the statements of § 2.1 have a pointwise meaning at those points where the entering maps are differentiable. If, e.g., g is differentiable at $q_1 \in N_1$, then $g^*_{q_1}A$ is well defined.

2.1.2.1. If N is of class $\mathscr{C}^1$, and $f: N \to M$ is continuous, then a form of bidegree $(0, 1)$ along f is simply a map $F: N \to TM$ with $\pi \circ F = f$ (π is the projection of TM onto M), i.e., a *vector field* along f. If F is continuous everywhere and differentiable at $q \in N$, then the *covariant differential* $(DF)_q$ is well defined as a linear map $T_qN \to T_{f(q)}M$ by requiring: (a) additivity, (b) the product rule for continuous coefficient functions which are differentiable at q, and

$$(2.1) \qquad (c): (Df^*X)_q u = \nabla_{f_{*q}u} X, \qquad u \in T_qN$$

for smooth vector fields X on M.

If $q = g(q_1)$, and g is differentiable at q_1, then $g^*F = F \circ g$ is differentiable at q_1 and $(D(g^*F))_{q_1} = g^*_{q_1}(DF)_q$.

2.1.3. In the case where $N = M$ and $f = \mathrm{id}$, one is led to the calculus of Flanders [8]. In particular, if $X_1, \ldots, X_m$ form a smooth positive orthonormal base field with dual forms $\sigma_1, \ldots, \sigma_m$, then

$$(2.2) \qquad DX_i = \sum_j \omega_{ji} X_j\,, \qquad d\omega_{ij} = \Omega_{ij} - \sum_k \omega_{ik} \wedge \omega_{kj}\,,$$

where ω_{ij}, Ω_{ij} are the *connection* and *curvature forms*. An example of a form of bidegree $(2, 2)$ is the *curvature operator*

$$(2.3) \qquad \mathscr{R} := \frac{1}{2} \sum_{i,j} \Omega_{ij} X_i \wedge X_j\,.$$

For even $m = 2(h + 1)$ the *Gauss-Bonnet-Chern form* γ on M is expressible by $\mathscr{R}$:

$$(2.4) \qquad \begin{aligned} \gamma &:= \frac{1}{(2\pi)^h \cdot h!}[\mathscr{R}^{h+1}] \\ &= \frac{1}{2^m \cdot \pi^h \cdot h!} \sum_{i_1 \cdots i_m} \varepsilon_{i_1 \cdots i_m} \Omega_{i_1 i_2} \wedge \cdots \wedge \Omega_{i_{m-1} i_m}\,. \end{aligned}$$

For m odd, one puts $\gamma := 0$.

2.2. The integration theory, appropriate for our purpose, can be found in Krickeberg [12]. We only presuppose it for forms defined on manifolds of class $\mathscr{C}^1$, and the measurability of a form shall include here its local boundedness, so that every measurable n-form ω on an oriented $\mathscr{C}^1$-manifold N has a finite integral $\int_\Delta \omega$ over any compact subset Δ of N. Relevant notions of the category of locally lipschitz manifolds, which is used in [12], are also given in [19]. We make some further statements as follows.

2.2.1. Let $N, \tilde{N}$ be $\mathscr{C}^1$-manifolds, $\tilde{N}$ oriented, and $g: \tilde{N} \to N$ locally lipschitz (by Rademacher's theorem [11, 11.1.3.1], g is then differentiable almost everywhere in $\tilde{N}$). For any continuous $\tilde{n}$-form ω on N, the $\tilde{n}$-form $g^*\omega$ is measurable. At the points $\tilde{q} \in \tilde{N}$ where g is not differentiable, we set $(g^*\omega)_{\tilde{q}} = 0$.

2.2.2. If $N, \tilde{N}$ are both compact and oriented and of the same dimension n, and $g: \tilde{N} \to N$ is a locally bilipschitz homeomorphism preserving orientation, then $g^*\omega$ is again measurable for any measurable n-form ω on N, and we have the transformation rule

$$\int_{\tilde{N}} g^*\omega = \int_N \omega \, . \tag{2.5}$$

For the local case cf. [11, 11.2.3.3].

2.2.3. Let $S \subset N$ be a compact oriented locally lipschitz submanifold of dimension $\tilde{n}$. Assume that S can be parametrized by a $\mathscr{C}^1$-manifold, i.e., that there are an oriented $\mathscr{C}^1$-manifold $\tilde{N}$ of dimension $\tilde{n}$ and a locally bilipschitz orientation-preserving homeomorphism $f: \tilde{N} \to S$. Then for any $\tilde{n}$-form ω on N of class $\mathscr{C}^0$ the integral $\int_{\tilde{N}} f^*\omega$ is independent of the parametrization $(f, \tilde{N})$ by (2.5), and hence one can set

$$\int_S \omega := \int_{\tilde{N}} f^*\omega \, . \tag{2.6}$$

It is in this sense that the integral in (3.17) will be understood.

3. The generalized Allendoerffer-Weil formula

3.1. For the riemannian manifold M the extended exterior calculus immediately leads to the infinitesimal version of the Gauss-Bonnet theorem, i.e., the Chern equation in T_1M [8]: Consider the inclusion $I: T_1M \subsetneq TM$ as a smooth vector field along the projection $\pi_1: T_1M \to M$, set $\mathscr{R}_1 := \pi_1^*\mathscr{R}$, and define differential forms A_k on T_1M by

$$A_k := [\mathscr{R}_1^k \wedge I \wedge (DI)^{m-2k-1}] \, , \qquad 0 \le k \le [\tfrac{1}{2}(m-1)] \, . \tag{3.1}$$

Then the Chern equation can be stated in the form

(3.2) $$\gamma_1 := \pi_1^* \gamma = -d\Pi \qquad \text{with } \Pi := \sum_{k=0}^{h} a_k A_k ,$$

where

(3.3) $$h := [\tfrac{1}{2}(m-1)] ,$$
$$a_k := \begin{cases} [(2\pi)^{h+1} \cdot k! \cdot 1 \cdot 3 \cdots (m-2k-1)]^{-1} & \text{for } m \text{ even} , \\ [2 \cdot (2\pi)^h \cdot k! \cdot 2 \cdot 4 \cdots (m-2k-1)]^{-1} & \text{for } m \text{ odd} . \end{cases}$$

For any map $F: N \to TM$ we put (in the complement of the zeros of F) $F^0 := |F|^{-1} \cdot F$.

3.2. Lemma. *Let Y be a locally lipschitz vector field on M.*

(i) *If $\Delta \subset M$ is a compact $\mathscr{C}^1$-piece* [2] *containing no zeros of Y, then*

(3.4) $$\int_\Delta \gamma = -\int_{\partial\Delta} (Y^0 | \partial\Delta)^* \Pi .$$

(ii) *If Y is of class $\mathscr{C}^\infty$ around a regular zero p_0, then*

(3.5) $$\lim_{\Gamma \to p_0} \int_{\partial\Gamma} (Y^0 | \partial\Gamma)^* \Pi = \operatorname{ind}_{p_0} Y \in \{-1, 1\}$$

for $\mathscr{C}^\infty$-pieces Γ contracting to p_0.

Proof. (i) follows by applying Krickeberg's version of the Stokes' theorem [12, § 9] to the equation $\gamma = -Y^{0*} d\Pi$ which holds by (3.2) almost everywhere in a neighborhood of Δ. (ii) is well known [5].

3.3. Lemma. *Let $\Gamma \subset M$ be a compact $\mathscr{C}^\infty$-piece. Then the integral*

(3.6) $$\int_{\partial\Gamma} V^{0*} \Pi$$

has the same value for any locally lipschitz vector field $V: \partial\Gamma \to TM$ which is exterior to Γ.

Proof. Construct a sequence of compact $\mathscr{C}^\infty$-pieces $\Gamma_j \subset \Gamma^0$ such that

(3.7) $$\partial(\Gamma \backslash \Gamma_j^0) = \partial\Gamma \cup \partial\Gamma_j , \qquad \operatorname{vol}(\Gamma \backslash \Gamma_j^0) \to 0 ,$$

and that a field ν_j of outer normal vectors along $\partial\Gamma_j$ and V can be continued to a locally lipschitz vector field Y which has no zeros in a neighorhood of $\Gamma \backslash \Gamma_j^0$ [15, 5.9]. Then by Lemma 3.2 (i) we have

(3.8) $$\int_{\Gamma \backslash \Gamma_j^0} \gamma = -\int_{\partial\Gamma} V^{0*} \Pi + \int_{\partial\Gamma_j} \nu_j^{0*} \Pi .$$

Here the terms depending on j converge with limits independent of V.

3.4. Lemma. *Let $\Delta \subset M$ be a compact $\mathscr{C}^1$-piece, and Y a continuous vector field on M, exterior to Δ along $\partial\Delta$. Then there exists a compact $\mathscr{C}^\infty$-piece $\Gamma \subset M$ such that*

(i) $\Gamma \subset \Delta^0$,

(ii) *Y is exterior to Γ along $\partial\Gamma$,*

(iii) *$\Delta \backslash \Gamma^0$ contains no zeros of Y,*

(iv) *Δ and Γ are $\mathscr{C}^1$-diffeomorphic.*

Proof. Let $\lambda = (\lambda_1, \lambda_2)$ be a $\mathscr{C}^1$-diffeomorphism of a neighborhood W of $\partial\Delta$ onto $\partial\Delta \times]-1, 1[$ transforming $W \cap \Delta$ onto $\partial\Delta \times]-1, 0]$; see [15, 5.9]. Consider the continuous vector field Λ on W associated with $\partial/\partial t$ on $\partial\Delta \times]-1, 1[$. One can assume $Y \neq 0$, and

$$\psi_1 := \frac{\pi}{2} - \sphericalangle(\text{grad}\, \lambda_2, \Lambda) > 0 \,, \tag{3.9}$$

$$\psi_2 := \frac{\pi}{2} - \sphericalangle(\text{grad}\, \lambda_2, Y) > 0 \tag{3.10}$$

on W. Put $\psi := \min\{\psi_1, \psi_2\}$. Choose t_1, t_1' with $-1 < t_1 < t_1' < 0$ and, according to [15, 4.2], a smooth function μ on W such that

$$|\mu - \lambda_2| < \tfrac{1}{2}(t_1' - t_1) \,, \tag{3.11}$$

$$|\text{grad}\, \mu - \text{grad}\, \lambda_2| < \tan^2(\tfrac{1}{2}\psi) \cdot |\text{grad}\, \lambda_2| \,. \tag{3.12}$$

Then

$$\Gamma := \Delta \backslash \{p \in W \,|\, \mu(p) > \tfrac{1}{2}(t_1' + t_1)\} \tag{3.13}$$

will satisfy the assertions. (Observe that (3.12) implies $\sphericalangle(\text{grad}\, \mu, \text{grad}\, \lambda_2) < \psi$, and that $t \mapsto \mu \circ \lambda^{-1}(p, t)$ is strictly increasing for every $p \in \partial\Delta$, assuming the value $\frac{1}{2}(t_1' + t_1)$ exactly once.)

3.5. Proposition. *Let $\Delta \subset M$ be a compact $\mathscr{C}^1$-piece, and V an exterior locally lipschitz vector field on $\partial\Delta$. Then*

$$\int_\Delta \gamma = -\int_{\partial\Delta} V^{0*}\Pi + \chi(\Delta) \,, \tag{3.14}$$

where χ denotes the Euler-Poincaré characteristic.

Proof. Let Y be a locally lipschitz continuation of V onto M. Choose Γ according to § 3.4. Denoting the exterior unit normal vector field along $\partial\Gamma$ by ν, we have

$$\int_\Gamma \gamma = -\int_{\partial\Delta} \nu^*\Pi + \chi(\Gamma) \,. \tag{3.15}$$

This is the Allendoerffer-Weil formula particularly for the smooth polyhedron Γ. (We can also obtain an independent proof (3.15) by using a smooth Morse function, whose gradient is exterior to Γ along $\partial\Gamma$, and applying § 3.2 and the smooth version of the Hopf (or Morse) index theorem.) Now by (3.15), and Lemmas 3.2 (i), 3.3 and 3.4 we have

$$\begin{aligned}\int_{\Delta} \gamma &= \int_{\Gamma} \gamma + \int_{\Delta\setminus\Gamma^0} \gamma \\ &= -\int_{\partial\Gamma} \nu^*\Pi + \chi(\Gamma) - \int_{\partial\Delta} V^{0*}\Pi + \int_{\partial\Gamma} (Y^0|\partial\Gamma)^*\Pi \qquad (3.16)\\ &= -\int_{\partial\Delta} V^{0*}\Pi + \chi(\Delta)\ .\end{aligned}$$

3.6.1. Let C always be a compact locally convex subset of M. The following statements are necessary for the following main formula; they are based on the results of [18]. In a suitable open neighborhood U of C we have a well-defined metric projection f onto C (previously denoted by *) which is locally lipschitz. For every $q \in U\setminus C$ we denote by $F(q)$ the unit initial vector of the geodesic normal $f(q)q$. F is a locally lipschitz vector field along f. Both maps are occasionally restricted to a fixed outer parallel hypersurface ∂C^r. These hypersurfaces are of class $\mathscr{C}^1$ for small r. The images of f and F are respectively the set theoretic boundary ∂C of C and the set $\mathscr{N}_C$ of unit outer vectors along ∂C. $\mathscr{N}_C$ is a compact locally lipschitz submanifold of T_1M of dimension $m - 1$ which is $\mathscr{C}^1$-parametrized by $F|\partial C^r$ (§ 2.2.3).

3.6.2. Theorem. *For any compact locally convex subset C of an oriented riemannian manifold M we have*

$$\int_C \gamma = -\sum_{k=0}^{h} a_k \int_{\mathscr{N}_C} [\mathscr{R}_1^k \wedge I \wedge (DI)^{m-2k-1}] + \chi(C)\ , \qquad (3.17)$$

where $\chi(C)$ is the Euler-Poincaré characteristic of C, and all quantities exist and are finite. Each integral on the right hand side can be written as

$$\int_{\partial C^r} [f^*\mathscr{R}^k \wedge F \wedge (DF)^{m-2k-1}] \qquad (3.18)$$

for any sufficiently small $r > 0$.

Proof. Since C is a strong deformation retract of each outer parallel set C^r,

$$\chi(C) = \chi(C^r) \qquad (3.19)$$

for small r. Consider the distance function ρ_C from C which is of class $\mathscr{C}^1$ in $U\setminus C$; its gradient V_C is of norm 1 and again locally lipschitz. First applying Proposition 3.5 to C^r and V_C and using (3.19) we obtain

$$\int_{C^r} \gamma = -\int_{\partial C^r} (V_C | \partial C^r)^* \Pi + \chi(C) . \tag{3.20}$$

Then take the limit as $r \downarrow 0$. Of course, the left hand side converges to $\int_C \gamma$ (which vanishes if C has m-measure 0). For handling the second term in (3.20) we have to transform it to a fixed domain of integration. Let $h_t, 0 < t \leq 1$, be the dilatations with center C defined in U, i.e., for $q \in U$, $h_t(q)$ is the unique point on $qf(q)$ with $\rho(q, h_t(q)) = (1 - t) \cdot \rho(q, f(q))$, where each h_t is a locally bilipschitz homeomorphism onto some open neighborhood of C; for details see [19]. Put $H_t := V_C \circ h_t$. In the terms of [18]:

$$h_t(q) = \exp(t \cdot \Phi(f(q), q)) , \tag{3.21}$$

$$H_t(q) = \frac{1}{\rho_C(q)} \cdot \Omega(\Phi(f(q), q), t) , \tag{3.22}$$

$$F(q) = \frac{1}{\rho_C(q)} \cdot \Phi(f(q), q) . \tag{3.23}$$

Denote, for a fixed r_0 and r with $0 < r \leq r_0$, by f_r, F_r respectively the restrictions of $h_{r/r_0}, H_{r/r_0}$ onto ∂C^{r_0}. Likewise, consider f, F only on ∂C^{r_0}. One can prove that the locally bilipschitz homeomorphism $f_r : \partial C^{r_0} \to \partial C^r$ preserves orientation [19], so by § 2.2.2 we have

$$\begin{aligned} \int_{\partial C^r} (V_C | \partial C^r)^* \Pi &= \int_{\partial C^{r_0}} f_r^* (V_C | \partial C^r)^* \Pi \\ &= \int_{\partial C^{r_0}} (V_C | \partial C^r \circ f_r)^* \Pi = \int_{\partial C^{r_0}} F_r^* \Pi . \end{aligned} \tag{3.24}$$

Now (3.22), (3.23) show that

$$\lim_{r \downarrow 0} (F_r)_{*q} u = F_{*q} u , \qquad u \in T_q(\partial C^{r_0}) \tag{3.25}$$

for any $q \in \partial C^{r_0}$, where f is differentiable, i.e., up to a set of $(m - 1)$-measure 0 in ∂C^{r_0}. Thus Lebesgue's pointwise convergence theorem gives, in consequence of (3.24),

$$\lim_{r \downarrow 0} \int_{\partial C^r} (V_C | \partial C^r)^* \Pi = \int_{\partial C^{r_0}} F^* \Pi , \tag{3.26}$$

since the uniform boundedness condition is also fullfilled by the local lipschitz property of f and the compactness of ∂C^{r_0}.

A pointwise calculation shows

$$\begin{aligned} F^* \Pi &= \sum_{k=0}^{h} a_k F^* [\mathscr{R}_1^k \wedge I \wedge (DI)^{m-2k-1}] \\ &= \sum_{k=0}^{h} a_k [f^* \mathscr{R}^k \wedge F \wedge (DF)^{m-2k-1}] \end{aligned} \tag{3.27}$$

almost everywhere on ∂C^{r_0}. Thus the assertion follows from (2.6), (3.20), (3.26) and § 2.2.3. (The orientation of $\mathcal{N}_C$ is the one induced by $F\,|\,\partial C^{r_0}$, and does not depend on the choice of r_0.)

3.7.1. For applications it is useful to represent the boundary terms of (3.18) more explicitely in an adapted base. In particular, the resulting formula will contain the coefficients of the generalized second fundamental form II of ∂C [19]. The form II is defined almost everywhere by

$$\mathrm{II} = \langle df \otimes DF \rangle \,, \tag{3.28}$$

where f, F can either be considered on $U \backslash C$ or on any fixed parallel hypersurface ∂C^r (r small). In any case, II is symmetric and positive semidefinite almost everywhere.

3.7.2. Again, restrict f, F on ∂C^{r_0}, and let them be differentiable at q_0. Choose a smooth positive orthonormal base field $X_1, \cdots, X_m$ in a neighborhood of $p_0 := f(q_0)$ such that

$$\begin{aligned} &X_m\,|_{p_0} = F\,|_{q_0} \,, \\ &X_1\,|_{p_0}, \cdots, X_d\,|_{p_0} \text{ span } (df)_{q_0}(T_{q_0}(\partial C^{r_0})) \,, \end{aligned} \tag{3.29}$$

$d\,(= d_{q_0})$ being the rank of f at q_0 (in general, d_{q_0} is not continuously dependent on q_0). We fix the following ranges of indices:

$$\begin{aligned} &1 \le \alpha, \beta, \cdots \le d \,, \qquad d+1 \le A, B, \cdots \le m \,, \\ &1 \le i, j, \cdots \le m \,. \end{aligned} \tag{3.30}$$

Then

$$(df)_{q_0} = \sum_\alpha (f^*\sigma_\alpha)_{q_0} X_\alpha\,|_{q_0} \,, \qquad (f^*\sigma_A)_{q_0} = 0 \,. \tag{3.31}$$

$(DF)_{q_0}$ decomposes according to

$$(DF)_{q_0} = \sum_\alpha \pi_\alpha \cdot X_\alpha\,|_{p_0} + \sum_A \pi_A \cdot X_A\,|_{p_0} =: G + H \,, \tag{3.32}$$

where the π_i are linear forms on $T_{q_0}(\partial C^{r_0})$ and

$$\pi_m = 0 \,, \tag{3.33}$$

since $\langle F, F \rangle = 1$. The second fundamental form is

$$\mathrm{II}_{q_0} = \langle (df)_{q_0} \otimes (DF)_{q_0} \rangle = \sum_\alpha (f^*\sigma_\alpha)_{q_0} \otimes \pi_\alpha \,, \tag{3.34}$$

and since it is symmetric there is a symmetric matrix $(b_{\alpha\beta})$ such that

$$\pi_\alpha = \sum_\beta b_{\alpha\beta} (f^*\alpha_\beta)_{q_0} \,. \tag{3.35}$$

Note

$$(f^*\mathscr{R})_{q_0} = \tfrac{1}{2}\sum_{i,j}(f^*\Omega_{ij})_{q_0}(X_i \wedge X_j)_{p_0}\,, \tag{3.36}$$

where

$$\Omega_{ij} = \tfrac{1}{2}\sum_{k,l} R_{ijkl}\sigma_k \wedge \sigma_l\,. \tag{3.37}$$

This gives

$$\begin{aligned}&[f^*\mathscr{R}^k \wedge F \wedge (DF)^{m-2k-1}]_{q_0}\\ &\quad = \sum_{\mu=0}^{m-2k-1}\binom{m-2k-1}{\mu}[(f^*\mathscr{R}^k)_{q_0} \wedge F|_{q_0} \wedge G^{m-2k-1-\mu} \wedge H^{\mu}]\,.\end{aligned} \tag{3.38}$$

The indices $A, B, \cdots$ can now be restricted to $d+1, \cdots, m-1$. Use of (3.29), (3.32), (3.36) yields

$$\begin{aligned}&[(f^*\mathscr{R}^k)_{q_0} \wedge F|_{q_0} \wedge G^{m-2k-1-\mu} \wedge H^{\mu}]\\ &\quad = (-1)^{m-1}\cdot 4^{-k}\cdot \sum \varepsilon_{i_1\cdots i_{2k}\alpha_{2k+1}\cdots\alpha_{m-\mu-1}A_{m-\mu}\cdots A_{m-1}}\\ &\qquad \cdot R_{i_1i_2\beta_1\beta_2}(p_0)\cdots R_{i_{2k-1}i_{2k}\beta_{2k-1}\beta_{2k}}(p_0)\\ &\qquad \cdot b_{\alpha_{2k+1}\beta_{2k+1}}\cdots b_{\alpha_{m-\mu-1}\beta_{m-\mu-1}}\\ &\qquad \cdot (f^*\sigma_{\beta_1}\wedge\cdots\wedge f^*\sigma_{\beta_{m-\mu-1}})_{q_0} \wedge \pi_{A_{m-\mu}} \wedge\cdots\wedge \pi_{A_{m-1}}\,,\end{aligned} \tag{3.39}$$

the sums being taken over all repeated indices. Here all terms with $m-\mu-1 \neq d$ will vanish, so

$$\begin{aligned}&[f^*\mathscr{R}^k \wedge F \wedge (DF)^{m-2k-1}]_{q_0}\\ &\quad = \binom{m-2k-1}{d-2k}\cdot[(f^*\mathscr{R}^k)_{q_0} \wedge F|_{q_0} \wedge G^{d-2k} \wedge H^{m-d-1}]\\ &\quad = 4^{-k}\cdot\binom{m-2k-1}{d-2k}\cdot\sum \varepsilon_{i_1\cdots i_{2k}\alpha_{2k+1}\cdots\alpha_d A_{d+1}\cdots A_{m-1}}\\ &\qquad \cdot \varepsilon_{\beta_1\cdots\beta_d A_{d+1}\cdots A_{m-1}}\\ &\qquad \cdot R_{i_1i_2\beta_1\beta_2}(p_0)\cdots R_{i_{2k-1}i_{2k}\beta_{2k-1}\beta_{2k}}(p_0)\\ &\qquad \cdot b_{\alpha_{2k+1}\beta_{2k+1}}\cdots b_{\alpha_d\beta_d}\cdot\mu_d\,,\end{aligned} \tag{3.40}$$

where $\binom{p}{q} = 0$ for $q < 0$, and

$$\mu_d := (-1)^{m-1}\cdot(f^*\sigma_1 \wedge\cdots\wedge f^*\sigma_d)_{q_0} \wedge \pi_{d+1} \wedge\cdots\wedge \pi_{m-1}\,. \tag{3.41}$$

Finally, the $i_1, \cdots, i_{2k}$ must be $\leq d$, say $i_1 = \alpha_1, \cdots, i_{2k} = \alpha_{2k}$, and then the sign factors in (3.40) collapse to $\varepsilon_{\alpha_1\cdots\alpha_d}\cdot\varepsilon_{\beta_1\cdots\beta_d}$, so that (3.40) reduces to

$$[f^*\mathscr{R}^k \wedge F \wedge (DF)^{m-2k-1}]_{q_0}$$

$$= 4^{-k} \cdot \binom{m-2k-1}{d-2k} \cdot \sum \varepsilon_{\alpha_1 \cdots \alpha_d} \varepsilon_{\beta_1 \cdots \beta_d} \tag{3.42}$$

$$\cdot R_{\alpha_1 \alpha_2 \beta_1 \beta_2}(p_0) \cdots R_{\alpha_{2k-1} \alpha_{2k} \beta_{2k-1} \beta_{2k}}(p_0)$$

$$\cdot b_{\alpha_{2k+1} \beta_{2k+1}} \cdots b_{\alpha_d \beta_d} \cdot \mu_d \,.$$

The coefficient of μ_d has the same form as in Allendoerffer-Weil's formula.

4. A Cohn-Vossen inequality for geodesically convex sets

4.1.1. If the curvature satisfies positivity conditions, then the boundary terms in Theorem 3.6.2 can be estimated. But first we have the following lemma on the form μ_d given by (3.41) which is intrinsically defined because it agrees for two positive othonormal bases satisfying (3.29). Observe that μ_d is not, as one could expect, the volume form of the metric induced from the bundle metric of T_1M by F.

4.1.2. Lemma. *If the metric projection f is differentiable at $q_0 \in \partial C^{r_0}$, and $(df)_{q_0}$ has rank d, then μ_d given by* (3.41) *is positive in the canonical orientation of ∂C^{r_0}.*

Proof. $\mu_d \neq 0$ follows from (3.31), (3.35) and the fact that $F : \partial C^{r_0} \to TM$ is of rank $m-1$, [19]. To show $\mu_d \geq 0$ let $u_1, \cdots, u_{m-1}$ be a positive basis of $T_{q_0}(\partial C^{r_0})$ such that the u_A span $\ker f_{*q_0}$. Since $f_r : \partial C^{r_0} \to \partial C^r$ preserves orientation for any $0 < r \leq r_0$,

$$f_{r*q_0} u_1 \wedge \cdots \wedge f_{r*q_0} u_{m-1} > 0 \tag{4.1}$$

in $T_{f_r(q_0)}(\partial C^r)$. Let $\bar{f}_r$ be the geodesic dilatation of ratio r/r_0 with center $p_0 := f(q_0)$, i.e., let

$$\bar{f}_r(q) = \exp\left(\frac{r}{r_0} \Phi(p_0, q)\right) . \tag{4.2}$$

On $T_{q_0}(S_{r_0}(p_0))$ we have

$$\lim_{r \downarrow 0} \frac{1}{r} \bar{f}_{r*q_0} = \left(D\left(\frac{1}{r_0} \Phi(p_0, \mathrm{id})\right)\right)_{q_0} . \tag{4.3}$$

Now from $f_{*q_0} u_A = 0$ follow

$$f_{r*q_0} u_A = \bar{f}_{r*q_0} u_A \,, \tag{4.4}$$

$$\left(D\left(\frac{1}{r_0} \Phi(p_0, \mathrm{id})\right)\right)_{q_0} u_A = (DF)_{q_0} u_A \,. \tag{4.5}$$

This implies, in consequence of (4.1),

$$(df)_{q_0}u_1 \wedge \cdots \wedge (df)_{q_0}u_d \wedge (DF)_{q_0}u_{d+1} \wedge \cdots \wedge (DF)_{q_0}u_{m-1} \geq 0 \tag{4.6}$$

in $(F_{q_0})^{\perp} \subset T_{q_0}M$. Substituting (3.31), (3.32) for df, DF in (4.6) shows that μ_d has nonnegative value on $u_1, \cdots, u_{m-1}$.

4.2.1. The curvature operator $\mathscr{R}_p$ given by (2.3), $p \in M$, defines a linear map of $\bigwedge^2 T_pM$ into itself by requiring $\mathscr{R}_p(x \wedge y) = \mathscr{R}_p(x, y)$ for all decomposable $x \wedge y \in \bigwedge^2 T_pM$. If $\langle\!\langle\ ,\ \rangle\!\rangle_p$ denotes the natural scalar product on $\bigwedge^2 T_pM$, then $\mathscr{R}_p$ corresponds to the bilinear form

$$R_p(\underline{X}, \underline{Y}) := \langle\!\langle \mathscr{R}_p\underline{X}, \underline{Y} \rangle\!\rangle_p\ , \qquad \underline{X}, \underline{Y} \in \textstyle\bigwedge^2 T_pM\ . \tag{4.7}$$

In an orthonormal base,

$$R_p(X_i \wedge X_j, X_k \wedge X_l) = R_{ijkl}(p)\ . \tag{4.8}$$

The positive semidefiniteness of $\mathscr{R}_p$ means that R_p is nonnegative on the diagonal of $(\bigwedge^2 T_pM) \times (\bigwedge^2 T_pM)$, while nonnegativity of the sectional curvature K_p means that R_p is nonnegative on the diagonal of the subset of decomposable elements. The former implies the latter, but the converse is questionable for $m \geq 4$. B. Konstant proved (unpublished) that the positive semidefiniteness of $\mathscr{R}_p$ implies, in all even dimensions $m = 2(h+1)$, that the Gauss-Bonnet-Chern form (i.e., $[\mathscr{R}_p^{h+1}]$) is nonnegative (communicated by A. Weinstein). More generally, we have the following purely algebraic lemma (the notions are clear from above).

4.2.1.1. Lemma. *Let $(V, \langle\ ,\ \rangle)$ be an oriented euclidean vector space of dimension d, let $L: V \to V$ be linear, symmetric, and positive semidefinite with respect to $\langle\ ,\ \rangle$, and let $\mathscr{R}: \bigwedge^2 V \to \bigwedge^2 V$ be linear, symmetric, and positive semidefinite with respect to $\langle\!\langle\ ,\ \rangle\!\rangle$. Then, for $k, l \geq 0$ with $2k + l = d$, $[\mathscr{R}^k \wedge L^l]$ is a nonnegative d-form on V.*

Proof. We use similar symbols as above but without the subscript p; π_α and $b_{\alpha\beta}$ are now defined by

$$L(X) = \sum_\alpha \pi_\alpha(X)X_\alpha\ , \qquad \pi_\alpha = \sum_\beta b_{\alpha\beta}\sigma_\beta\ . \tag{4.9}$$

Then we have

$$\begin{aligned} 4^k \cdot [\mathscr{R}^k \wedge L^l] = \sum \varepsilon_{\alpha_1 \cdots \alpha_d}\varepsilon_{\beta_1 \cdots \beta_d} R_{\alpha_1\alpha_2\beta_1\beta_2} \cdots R_{\alpha_{2k-1}\alpha_{2k}\beta_{2k-1}\beta_{2k}} \\ \cdot b_{\alpha_{2k+1}\beta_{2k+1}} \cdots b_{\alpha_d\beta_d} \cdot \sigma_1 \wedge \cdots \wedge \sigma_d\ . \end{aligned} \tag{4.10}$$

By the assumption on L there is a positive orthonormal base in which $(b_{\alpha\beta})$

diagonalizes to $b_{\alpha\beta} = b_\alpha \delta_{\alpha\beta}$ with $b_\alpha \geq 0$. In this base, the coefficient of $\sigma_1 \wedge \cdots \wedge \sigma_d$ in (4.10) is of the form

$$\sum C_{\gamma_{2k+1}\cdots\gamma_d} \cdot b_{\gamma_{2k+1}} \cdots b_{\gamma_d} \tag{4.11}$$

with

$$C_{\gamma_{2k+1}\cdots\gamma_d} := \sum \varepsilon_{\alpha_1\cdots\alpha_{2k}\gamma_{2k+1}\cdots\gamma_d} \varepsilon_{\beta_1\cdots\beta_{2k}\gamma_{2k+1}\cdots\gamma_d} \cdot R_{\alpha_1\alpha_2\beta_1\beta_2} \cdots R_{\alpha_{2k-1}\alpha_{2k}\beta_{2k-1}\beta_{2k}} . \tag{4.12}$$

Thus it suffices to show that all $C_{\gamma_{2k+1}\cdots\gamma_d} \geq 0$. For this, consider the symmetric bilinear form R on $\bigwedge^2 V$ defined by (4.7). In a canonical way, R induces a unique symmetric bilinear form R_k on the tensor product $(\bigwedge^2 V) \otimes \cdots \otimes (\bigwedge^2 V)$ (k factors) such that

$$R_k(\underline{X}_1 \otimes \cdots \otimes \underline{X}_k, \underline{Y}_1 \otimes \cdots \otimes \underline{Y}_k) = R(\underline{X}_1, \underline{Y}_1) \cdots R(\underline{X}_k, \underline{Y}_k) \tag{4.13}$$

for all $\underline{X}_k, \underline{Y}_k \in \bigwedge^2 V$. The positive semidefiniteness of R extends to R_k. Now a calculation shows that (4.12) is just the value of R_k at (E, E) where

$$E := \sum \varepsilon_{\alpha_1\cdots\alpha_{2k}\gamma_{2k+1}\cdots\gamma_d} (X_{\alpha_1} \wedge X_{\alpha_2}) \otimes \cdots \otimes (X_{\alpha_{2k-1}} \wedge X_{\alpha_{2k}}) . \tag{4.14}$$

Hence the assertion.

4.2.1.1.1. Supplement. *For $k \leq 2$, the assumption "$\mathscr{R}$ positive semidefinite" can be replaced by "$\mathscr{R}$ positive semidefinite on the decomposable elements of $\bigwedge^2 V$".*

Proof. It follows from Chern [6] by writing (4.12) explicitly.

From (3.42) and Corollary 4.2.1.1 we finally obtain, observing $0 \leq 2k \leq d \leq m - 1$:

4.2.2. Theorem. *Let C be a compact locally convex subset of M. If* $\dim M \leq 6$, *and the sectional curvature of M is nonnegative over all points of ∂C, then*

$$\int_C \gamma \leq \chi(C) . \tag{4.15}$$

For $\dim M \geq 7$, (4.15) *holds if the curvature operator of M is positive semidefinite over all points of ∂C.*

4.2.2.1. Remark. In the second case, the assumption on the curvature operator could be replaced by what follows from it, namely, that the algebraic Hopf conjecture holds for all powers of $\mathscr{R}$ along ∂C.

4.2.2.2. Corollary. *For a 4-dimensional complete noncompact riemannian manifold with nonnegative sectional curvature, the total curvature exists and satisfies*

$$0 \leq \int_M \gamma \leq \chi(M) . \tag{4.16}$$

This generalizes the well known result of Chern and Milnor [6] to the non-compact case.

4.2.2.3. Corollary. *For a complete noncompact riemannian manifold of arbitrary dimension with positive semidefinite curvature operator, the total curvature exists and satisfies* (4.16).

The proofs of these corollaries follow by applying Theorem 4.2.2 to the sets of an increasing filtration of M by totally convex (in particular, locally convex) sets whose existence is a fundamental result of Cheeger and Gromoll's structure theory [3]. Their results also imply the existence of $\chi(M)$ if M is complete and has $K \geq 0$.

Corollaries 4.2.2.2 and 4.2.2.3 have been announced in a previous version of this paper (1971). In the meanwhile, similar results have been obtained by W. A. Poor [16] by using a globally defined geodesically convex function on M (cf. [3]) and an approximation theorem of Greene and Wu [9]. However, Theorem 4.2.2 cannot be deduced in this manner.

4.2.2.4. Corollary. *Let M be isometricly immersed as a hypersurface in a flat manifold $\tilde{M}$. If M is complete and noncompact and has nonnegative sectional curvature $K \geq 0$, then*

$$0 \leq \int_M \gamma \leq \chi(M) , \tag{4.17}$$

including the existence of all quantities.

For the case $\tilde{M} = \boldsymbol{R}^{m+1}$, see Wu [21]. The proof follows at once from the Gauss equation

$$R(X \wedge Y, W \wedge Z) = \langle B(X, W), B(Y, Z)\rangle - \langle B(X, Z), B(Y, W)\rangle , \tag{4.18}$$

where B is the second fundamental form of M in $\tilde{M}$. By combining this with Weinstein's result [20] we obtain, since $\chi(M) = 1$ if $K > 0$ and M open [10],

4.2.2.5. Corollary. *Let M be isometricly immersed with codimension* 2 *in a flat manifold $\tilde{M}$. If M is complete and noncompact and has strictly positive sectional curvature $K > 0$, then the total curvature exists and satisfies*

$$0 \leq \int_M \gamma \leq 1 . \tag{4.19}$$

References

[1] C. B. Allendoerffer & A. Weil, *The Gauss-Bonnet theorem for riemannian polyhedra*, *Trans. Amer. Math. Soc.* **53** (1943) 101–129.

[2] N. Bourbaki, *Variétés différentielles et analytiques, Fasc. de rés.*, § 8 *à* 15, Elém. Math. Fasc. XXXVI, Paris 1971.

[3] J. Cheeger & D. Gromoll, *On the structure of complete manifolds of nonnegative curvature,* Ann. of Math. **96** (1972) 413–443.

[4] S. S. Chern, *A simple intrinsic proof of the Gauss-Bonnet formula for closed Riemannian manifolds,* Ann. of Math. **45** (1944) 747–752.

[5] ——, *On the curvatura integra in a Riemannian manifold,* Ann. of Moth. **46** (1945) 674–684.

[6] ——, *On curvature and characteristic classes of a Riemannian manifold,* Abh. Math. Sem. Univ. Hamburg **20** (1955) 117–126.

[7] S. Cohn-Vossen, *Kürzeste Wege und Totalkrümmung auf Flächen,* Comp. Math. **2** (1935) 69–133.

[8] H. Flanders, *Development of an extended exterior differential calculus,* Trans. Amer. Math. Soc. **75** (1953) 311–326.

[9] R. E. Greene & H. Wu, *On the subharmonicity and plurisubharmonicity of geodesically convex functions,* Indiana Univ. Math. J. **22** (1973) 642–653.

[10] D. Gromoll & W. Meyer, *On complete open manifolds of positive curvature,* Ann. of Math. **90** (1969) 75–90.

[11] O. Haupt, G. Aumann & C. Pauc, *Differential- und Integralrechnung, Bd.* III. *Integralrechnung,* Berlin, 1955.

[12] K. Krickeberg, *Über den Gaußschen und den Stokesschen Integralsatz.* II, Math. Nachr. **11** (1954) 35–60.

[13] N. H. Kuiper, *Der Satz von Gauss-Bonnet für Abbildungen in* E^N *und damit verwandte Probleme,* Jber. Deutsch. Math.-Verein, **69** (1967) 154–164.

[14] J. Milnor, *Topology from the differentiable viewpoint,* University Press of Virginia, Charlottesville, 1969.

[15] J. R. Munkres, *Elementary differential topology,* Annals of Math. Studies, No. 54, University of Princeton Press, Princeton, 1966.

[16] W. A. Poor, Jr., *Some results on nonnegatively curved manifolds,* Dissertation, State University of New York at Stony Brook, 1973, and J. Differential Geometry **9** (1974) 583–600.

[17] E. Portnoy, *Toward a generalized Gauss-Bonnet formula for complete, open manifolds,* Comment. Math. Helv. **46** (1971) 324–344.

[18] R. Walter, *On the metric projection onto convex sets in riemannian spaces,* Arch. Math. (Basel) **25** (1974) 91–98.

[19] ——, *Some analytical properties of geodesically convex sets,* to be published.

[20] A. Weinstein, *Positively curved* n-*manifolds in* $\mathbf{R}^{n+2}$, J. Differential Geometry **4** (1970) 1–4.

[21] H. Wu, *A structure theorem for complete noncompact hypersurfaces of nonnegative curvature,* Bull. Amer. Math. Soc. **77** (1971) 1070–1071.

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